RG for Ising

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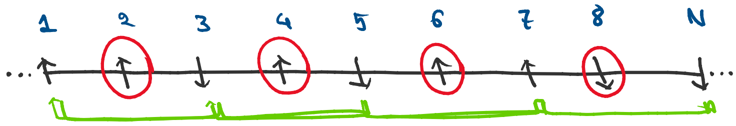
A coarse graining (decimation) in real space for 1d Ising model.

Consider an Ising on periodic chain : (zero magnetic field)

$$Z_N(J) = \sum_{S_1} \cdots \sum_{S_N} e^{J\sigma_1\sigma_2 + \cdots + J\sigma_N\sigma_1}$$



Decimate (sum over) all even spins in the partition sum.

For this we use

$$e^{J\sigma_1\sigma_2} = \cosh J\left[1 + (\tanh J)\,\sigma_1\sigma_2\right] = c_0\left(1 + v_0\sigma_1\sigma_2\right)$$

Then

$$e^{J\sigma_1\sigma_2 + J\sigma_2\sigma_3} = c_0^2\left(1 + v_0\sigma_1\sigma_2\right)\left(1 + v_0\sigma_2\sigma_3\right)$$

$$= c_0^2\left(1 + v_0(\sigma_1 + \sigma_3)\sigma_2 + v_0^2\,\sigma_1\sigma_2^2\sigma_3\right)$$

$$\Rightarrow \sum_{\sigma_2 = \pm 1} e^{J(\sigma_1\sigma_2 + \sigma_2\sigma_3)} = c_0^2\left(2 + 2v_0^2\,\sigma_1\sigma_3\right)$$

$$= 2c_0^2\left(1 + v_1\sigma_1\sigma_3\right)$$

$$= \frac{2c_0^2}{c_1}\cdot c_1\left(1 + v_1\sigma_1\sigma_3\right)$$

$$= \frac{2c_0^2}{c_1}\cdot e^{J_1\sigma_1\sigma_3}$$

$v_1 = v_0^2$

write

$v_1 = \tanh J_1$

$c_1 = \cosh J_1$

Performing the same for all even spins

$$Z_N(J_0) = \left(\frac{2c_0^2}{c_1}\right)^{\frac{N}{2}} \sum_{\substack{\text{odd}\\ \text{spins}}} e^{J_1\sigma_1\sigma_3 + \cdots + J_1\sigma_{N-1}\sigma_1}$$

$$= \left(\frac{2c_0^2}{c_1}\right)^{\frac{N}{2}} Z_{\frac{N}{2}}(J_1)$$

$\longleftarrow$ Partition function of another Ising chain of $\frac{N}{2}$ spins

Writing for free energy densities

$$N f(J_0) = -\frac{N}{2}\log\left(\frac{2c_0^2}{c_1}\right) + \frac{N}{2} f(J_1) \qquad \text{for large } N$$

$$\Rightarrow \boxed{f(J_1) = f_0 + 2f(J_0)} \quad \text{with} \quad \boxed{\tanh J_1 = (\tanh J_0)^2}$$

Important points

(1) It is the same Ising model (Hamiltonian) ie same function $f(J)$.

(2) Decimation step is equivalent to coarse graining, or looking at a different length scale. For correlation length this means

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(less resolved microscope!)

$$\boxed{\xi(J_0) = 2\,\xi(J_1)}$$

(3) The decimation could be by b blocks in general. In that case we get

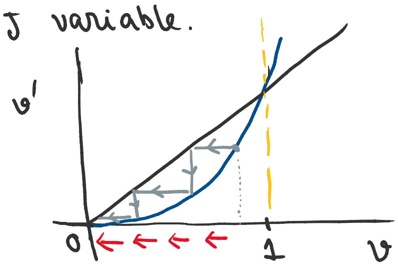
$$f(J_1) = f_0 + b\, f(J_0) \quad \text{and} \quad \tanh J_1 = (\tanh J_0)^b$$

and

$$\xi(J_1) = \frac{1}{b}\,\xi(J_0)$$

Repeating the decimation procedure, there is a flow of free energy densities. One convenient way to see the flow is in $v = \tanh J$ variable.

In every decimation step $v' = v^b$



$v = 1$ is a repulsive fixed point.

$v = 0$ is an attractive fixed point.

$v = 0 \Rightarrow \tanh J = 0 \Rightarrow J = 0$ (or infinite temperature)

$v = 1 \Rightarrow \tanh J = 1 \Rightarrow J \to \infty$ (or zero temperature)

What this mean?

verbaly this means, more we look at a macroscopic (coarse grain) scale, the 1d Ising model behave more like in a reduced coupling strength.

This is the case for any finite J.

This then means, any finite J on a macroscopic scale behave like in a $J=0$ $(T\to\infty)$ phase. This does not apply for $J\to\infty$ $(T=0)$ Ising model.

This confirms that 1d Ising has no phase transition at non-zero temp.

[a phase transition would mean ]

For correlation length

$$\xi(v' = v^b) = \frac{1}{b}\,\xi(v)$$

$$\Rightarrow \xi(v) = \frac{-1}{\log(v)} \Rightarrow \boxed{\xi_J = \frac{-1}{\log \tanh J}}$$

This is the exact formula one gets from exact solution!

This formula for ξ_J means correlation length remains finite at any finite J.

For $J \to \infty$ (ie $T \to 0$): The above formula for correlation length gives

$$\xi_J \sim e^{J/k_B T}$$

which diverges exponentially as $T \to 0$ (not as power-law)

The case $h \neq 0$ (Ising in presence of magnetic field).

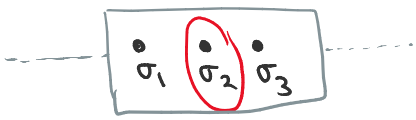
[Ref: see Kardar book, vol 2]

Original Hamiltonian

$$H = -J_0 \sum_i \sigma_i \sigma_{i+1} - h_0 \sum_i \sigma_i$$

$$= -J_0 \sum_i \sigma_i \sigma_{i+1} - \frac{h_0}{2} \sum_i (\sigma_i + \sigma_{i+1})$$

We want to write



$$\sum_{\sigma_2 = \pm 1} e^{J_0 \sigma_1 \sigma_2 + J_0 \sigma_2 \sigma_3 + \frac{h_0}{2}(\sigma_1 + \sigma_2) + \frac{h_0}{2}(\sigma_2 + \sigma_3)}$$

$$= e^{J_1 \sigma_1 \sigma_3 + \frac{h_1}{2}(\sigma_1 + \sigma_3) + g_1}$$

To get relations between (J_0, h_0, g_0) and (J_1, h_1, g_1) we define new variables

$$x_0 = e^{J_0}, \quad y_0 = e^{h_0}, \quad z_0 = e^{g_0} \qquad (\text{for us } g_0 = 0)$$

$$x_1 = e^{J_1}, \quad y_1 = e^{h_1}, \quad z_1 = e^{g_1}$$

This relation must be true for all possible configs of (σ_1, σ_3). This can be solved.

We get

$$y_1^2 = y_0^2 \, \frac{x_0^2 y_0 + x_0^{-2} y_0^{-1}}{x_0^{-2} y_0 + x_0^2 y_0^{-1}}$$

$$x_1^4 = \frac{(x_0^2 y_0 + x_0^{-2} y_0^{-1})(x_0^{-2} y_0 + x_0^2 y_0^{-1})}{(y_0 + y_0^{-1})^2}$$

does not involve z_1, z_0

$$z_1^4 = z_0^8 \, (x_0^2 y_0 + x_0^{-2} y_0^{-1})(x_0^{-2} y_0 + x_0^2 y_0^{-1})(y_0 + y_0^{-1})^2$$

This in terms of original variables mean

$$\boxed{\begin{aligned} J_1 &= J_1(J_0, h_0) \\ h_1 &= h_0 + \delta h(J_0, h_0) \end{aligned}} \quad \text{and} \quad \boxed{g_1 = 2g_0 + \delta g(J_0, h_0)}$$

The rescaled J, h do not involve g_0 $\Rightarrow$ g goes into additive constant of new Hamiltonian, and thereby do not contribute to singular part of free energy.

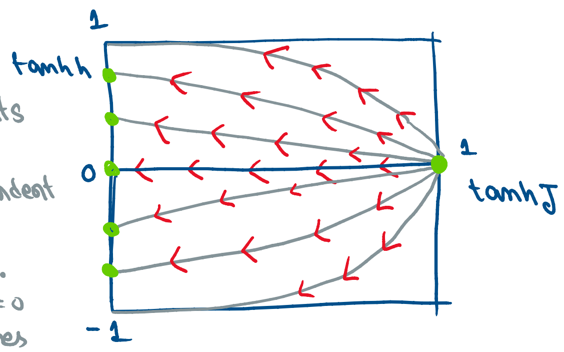
The new Hamiltonian

$$H_1 = -J_1 \sum_{\text{odd}} \sigma_i \sigma_{i+2} - h_1 \sum_{\text{odd}} \sigma_i + g_1$$

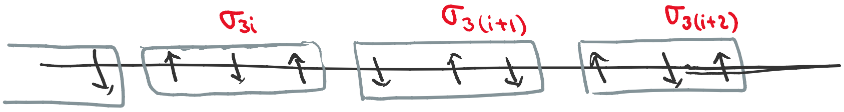
again another Ising Hamiltonian.

Flow chart

A line of fixed points at $J=0$ and they correspond to independent spins and zero correlation length. (see this by setting $x=0$ in the RG eq, gives $y'=y$)



Decimation step :



This corresponds to $b=3$

The decimation procedure coarse-grains and replaces three spins by one

$$\{\sigma_{3i-1}, \sigma_{3i}, \sigma_{3i+1}\} \longrightarrow \hat{\sigma}_{3i}$$

We did it by summing over $\{\sigma_{3i-1}, \sigma_{3i+1}\}$. There are many other alternatives.

For example $\quad \hat{\sigma}_{3i} = \mathrm{Sgn}\{\sigma_{3i-1} + \sigma_{3i} + \sigma_{3i+1}\}$

In all these decimation procedures, the basic idea is

$$Z = \sum_{\text{all } \sigma_i} e^{-\beta H[\sigma]} = \sum_{\hat{\sigma}} e^{-\beta \hat{H}[\hat{\sigma}]}$$

where $\hat{H}[\hat{\sigma}]$ is the new Hamiltonian that emerged from coarse graining procedure. This new Hamiltonian may have additional interactions.

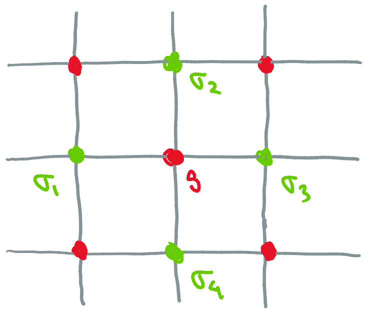
For the 1d-example, we saw

$$H = J_0 \sigma \sigma' + \frac{h_0}{2}(\sigma + \sigma') \longrightarrow \hat{H} = J_1 \hat{\sigma} \hat{\sigma}' + \frac{h_1}{2}(\hat{\sigma} + \hat{\sigma}') + g_1$$

Some of these new terms may survive, some may not. [this we discuss later]

In 1d, the decimation steps only generated a constant term g, which makes easy to follow the RG steps exactly.
In higher dimension, this is difficult, as decimation steps generate new interactions.

For example, square lattice Ising model. ($h=0$)



decimate the red sublattice spins

$$\sum_{s=\pm 1} e^{J_0 s(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)}$$

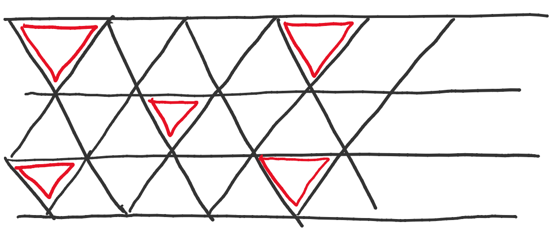
$$= e^{g_1 + J_1(\sigma_1\sigma_3 + \sigma_2\sigma_4) + J_1(\sigma_1\sigma_4 + \sigma_4\sigma_3 + \sigma_3\sigma_2 + \sigma_2\sigma_1) + \tilde{J}_2 \sigma_1\sigma_2\sigma_3\sigma_4}$$

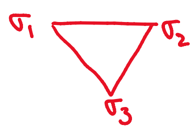
Repeating the decimation steps generate newer and newer interactions. More troublesome are the next nearest neighbor diagonal terms which appear with the same interaction strength as the nearest neighbor terms.

For some specific cases the RG procedure can be done in a perturbative way.

RG for Ising model on a triangular lattice

Niemeijer - van-Leeuwen method.





Decimation for each red plaquates

$$\hat{\sigma} = \text{sign}\{\sigma_1 + \sigma_2 + \sigma_3\}$$

The RG idea is to write a new effective Hamiltonian, such that

$$e^{-\beta \hat{H}[\hat{\sigma}]} = \sum_{\{\sigma\}}{}' e^{-\beta H[\sigma]}$$

the sum is done keeping the values of $\hat{\sigma}$ fixed.

It is very hard to carry out this restricted sum exactly. So the idea is to do it in a perturbative expansion.

$$H[\sigma] = H_0[\sigma] + V[\sigma]$$

contains interactions only insideeach red triangle. Therefor for this H_0 each red triangle are independent.

contains interactions between red triangles.

Then,

$$e^{-\hat{H}[\hat{\sigma}]} = \sum_{\sigma}{}' e^{-H_0[\sigma]} e^{-V[\sigma]} = \left[\sum_{\sigma}{}' e^{-H_0[\sigma]}\right] \left\{ \frac{\sum_{\sigma}' e^{-H_0 - V}}{\sum_{\sigma}' e^{-H_0[\sigma]}} \right\}$$

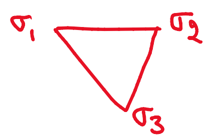
$$= Z_0[\hat{\sigma}] \times \langle e^{-V} \rangle_0 [\hat{\sigma}]$$

This gives the effective Hamiltonian

$$\hat{H}[\hat{\sigma}] = -\log Z_0[\hat{\sigma}] + \log \langle e^{-V} \rangle_0 [\hat{\sigma}]$$

The first term $Z_0[\hat{\sigma}]$. In each red triangle, for each value of new spins, there are four possibilities

$$h_0 = -J_0(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$



$\hat{\sigma}$	σ_1	σ_2	σ_3	e^{-h_0} ← for each triangle	
+	+	+	+	e^{3J_0}	their sum $(e^{3J_0} + 3e^{-J_0})$
+	−	+	+	e^{-J_0}	
+	+	−	+	e^{-J_0}	
+	+	+	−	e^{-J_0}	
−	−	−	−	e^{3J_0}	Their sum $(e^{3J_0} + 3e^{-J_0})$
−	+	−	−	e^{-J_0}	
−	−	+	−	e^{-J_0}	

	+	−	−	e
−	−	+	−	e^{-J_0}
−	−	−	+	e^{-J_0}

$\Big\}\ (e^{3J_0}+3e^{-J_0})$

We see that for each red triangle, the partion sum

$$z_0(\hat{\sigma}) = \sum_{\sigma} e^{-h_0(\sigma)} = e^{3J_0}+3e^{-J_0}$$ does not depend on $\hat{\sigma}$

$$\Rightarrow \quad Z_0[\hat{\sigma}] = \left(e^{3J_0}+3e^{-J_0}\right)^{N/3}$$ ← contributes a constant term g_1 in the effective Hamiltonian.

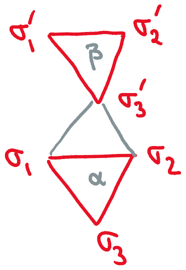
The second term $\langle e^{-V}\rangle_0$: this term $\log\langle e^{-V}\rangle_0[\hat{\sigma}]$ is calculated in a perturbation expansion method.

By definition

$$\log\langle e^{-V}\rangle_0 = -\langle V\rangle_0 + \frac{1}{2}\left[\langle V^2\rangle_0 - \langle V\rangle_0^2\right] + \cdots$$

cumulant expansion.

Linear term: $\langle V\rangle_0$ contains interactions between triangles.



$$\langle v\rangle_0[\hat{\sigma}_\alpha,\hat{\sigma}_\beta] = J_0\langle \sigma_3'\sigma_1\rangle_0 + J_0\langle \sigma_3'\sigma_2\rangle_0$$

$$= J_0\langle \sigma_3'\rangle_0\langle \sigma_1\rangle_0 + J_0\langle \sigma_3'\rangle_0\langle \sigma_2\rangle_0$$

Because H_0 is factorized in triangle plaquates.

From the top table we compute

$$\langle \sigma_1\rangle_0 = \langle \sigma_2\rangle_0 = \frac{e^{3J_0}+e^{-J_0}}{e^{3J_0}+3e^{-J_0}}\,\hat{\sigma}_\alpha$$

This gives us

$$\langle v\rangle_0[\hat{\sigma}_\alpha,\hat{\sigma}_\beta] = \underbrace{2J_0\left(\frac{e^{3J_0}+e^{-J_0}}{e^{3J_0}+3e^{-J_0}}\right)^2}\hat{\sigma}_\alpha\hat{\sigma}_\beta$$

and considering all plaquattes

$$\langle V\rangle_0[\hat{\sigma}] = J_1\sum_{\langle\alpha,\beta\rangle}\hat{\sigma}_\alpha\hat{\sigma}_\beta$$

And all together, the new Hamiltonian

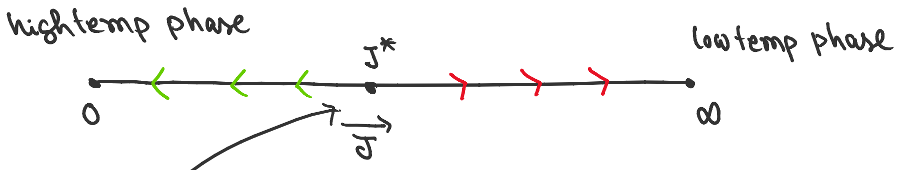
And all together, the new Hamiltonian

$$\hat{\mathcal{H}}[\hat{\sigma}] = -\frac{N}{3}\log\left(e^{3J_0}+3e^{-J_0}\right) - J_1\sum_{\langle\alpha,\beta\rangle}\hat{\sigma}_\alpha\hat{\sigma}_\beta + \mathcal{O}(V^2)$$

This, to leading order, corresponds to another triangular Ising model with a rescaled interaction strength

$$\boxed{J_1 = 2\cdot J_0\left(\frac{e^{3J_0}+e^{-J_0}}{e^{3J_0}+3e^{-J_0}}\right)^2}$$

The fixed point flow



unstable fixed point corresponds to critical point

$$J^* = 2J^*\left(\frac{e^{3J^*}+e^{-J^*}}{e^{3J^*}+3e^{-J^*}}\right)^2 \Rightarrow J^* \simeq 0.336$$

Exact result 0.2747

Remark: clearly the first order correction does not compare well with exact results. However, we can systematically include higher order terms which will introduce higher neighbor couplings. The flow diagram is to be constructed for this higher dimensional coupling space. In the original paper Niemeijer-van-Leeuwen PRL, 31 (1973) 1411 went to higher orders and the set of J^* they got are 0.336, 0.365, 0.255, 0.253, 0.281, 0.2741 which came closer to the exact value 0.2747.

Remark: an advantage of this "controlled" expansion method is that to each order the type & interactions generated in the decimation step is fixed.

The correlation length:

$$\xi(J_1) = \frac{1}{\sqrt{3}}\,\xi(J_0)$$

a linear stability analysis around J^* gives

$$\xi \sim \frac{1}{(\delta J)^{\nu}} \qquad \text{with } \nu = 1.133$$

[adding higher orders, NvL went as close as $\nu = 0.973$ to the exact result $\nu = 1$]

Free energy:

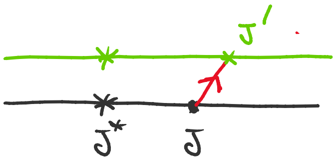
$$f(J_1) = \underset{b^2}{3}\, f(J_0) + f_0 \qquad \text{with } b=\sqrt{3}$$

Critical properties around the unstable fixed point J^*

Critical properties, particularly the power-law behavior are determines by the behavior of RG flow around the unstable fixed point. They can be estimated by linear stability analysis around this fixed point.

$$J' = 2J\left(\frac{e^{3J}+e^{-J}}{e^{3J}+3e^{-J}}\right)^2$$

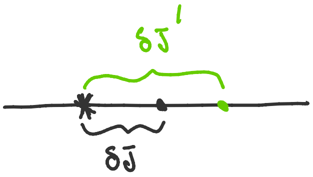
$$= 2J\left(\frac{1+e^{-4J}}{1+3e^{-4J}}\right)^2 = R(J) \quad \text{with} \quad J^* = R(J^*)$$



A decimation step at a point J near J^*.

$$J' = R(J) = R(J-J^*+J^*)$$

$$= \underbrace{R(J^*)}_{J^*} + R'(J^*)(J-J^*) + \cdots$$



$$\Rightarrow \quad J'-J^* \simeq R'(J^*)(J-J^*) \quad \text{for} \quad J \simeq J^*$$

$$\Rightarrow \quad \boxed{\delta J' = R'(J^*)\,\delta J}$$

gives how the distance from J^* changes in a decimation step.

Useful facts about $R'(J^*)$:

(1) $R'(J^*)$ depends on the decimation lengthscale $b=\sqrt{3}$ (for our case).

(2) It is standard to use a notation

$$\boxed{R'(J^*) = \lambda_b}$$

[we shall see later a connection to eigenvalue]

(3) Consider two successive decimation steps

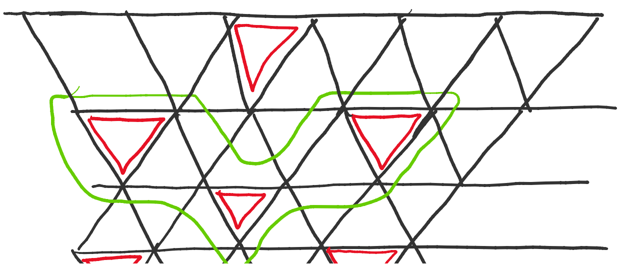
$$\delta J' = \lambda_b\,\delta J \quad \text{and} \quad \delta J'' = \lambda_b\,\delta J' = \lambda_b^2\,\delta J$$

This is also equivalent to a single decimation step by length b^2

$$\delta J'' = \lambda_{b^2}\,\delta J$$

[(a) note, this is within linear approximation.

(b) Canbe seen by counting how many sites been decimated



Repeating the steps

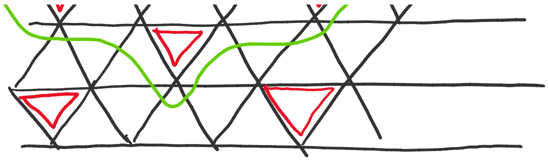
$$\lambda_{b^n} = (\lambda_b)^n$$

A non-trivial solution

$$\boxed{\lambda_b = b^y}$$

$$\boxed{\lambda_b = b^{y}}$$

[a general structure] [a number to be determined.]



(4) For our problem,

$$\lambda_b = b^{y} = R'(J^*) = 1.634 \qquad [\text{with } b = \sqrt{3}]$$

Critical exponents and intpretation of y:

(1) Correlation length:

$$\xi(J') = \frac{1}{b}\,\xi(J)$$

If we consider that correlation length diverge as power-law near critical point

i.e. $$\xi(J) \simeq \frac{\xi_0}{(J-J^*)^{\nu}}$$

then, $$\frac{\xi_0}{(J'-J^*)^{\nu}} = \frac{1}{b}\cdot\frac{\xi_0}{(J-J^*)^{\nu}} \Rightarrow (\delta J')^{\nu} = b\,(\delta J)^{\nu}$$

$$\Rightarrow b^{y\nu}\cdot(\delta J)^{\nu} = b(\delta J)^{\nu}$$

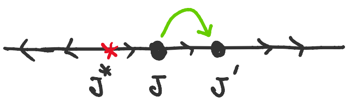
$$\Rightarrow \boxed{\nu = \frac{1}{y}}$$ the exponent y is inverse of the critical exponent ν.

(2) The free energy density.

$$f(J') = b^{d} f(J) + f_0$$

2 for our ease

comes from the constant term in new Hamiltoniant.



From critical phenomena we know that free energy density at a critical point is non-analytic.

$$f(J) = f_{na}(J) + \cdots$$

The $f_s(J)$ contains the leading non-analytic behavior

$$f_{na}(J) \sim (J-J^*)^{s}$$ with s being a positive non-integer number.

Combining with the decimation step

$$f_{na}(J') = b^{d} f_{na}(J) \Rightarrow (J'-J^*)^{s} = b^{d}(J-J^*)^{s}$$

$$\Rightarrow b^{ys}\cdot(\delta J)^{s} = b^{d}(\delta J)^{s}$$

$$\Rightarrow \boxed{\delta = d/y}$$

(3) Specific heat

$$C_v \sim -f''(J) \Rightarrow \text{near critical point}$$

$$C_v \sim -f''_{na}(J) \sim \frac{1}{(J-J^*)^{2-\delta}}$$

$$\Rightarrow C_v \sim \frac{1}{(J-J^*)^{\alpha}} \quad \text{with critical exponent} \quad \boxed{\alpha = 2 - \frac{d}{y}}$$

(4) Combining results for ν and α we get

$$\boxed{\alpha + d\nu = 2}$$ an example of scaling relation.

For other critical exponents we need to include non-zero magnetic field.

Presence of magnetic field $(h \neq 0)$

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

We follow an analysis very similar to described above.

$$H[\sigma] = H_0[\sigma] + V[\sigma] + V_h[\sigma] \rightarrow \text{contains } h\sum_i \sigma_i$$

$H_0[\sigma]$: contains interactions only inside each red triangle.

$V[\sigma]$: contains interactions between red triangles.

This gives the effective Hamiltonian

$$\hat{H}[\hat{\sigma}] = -\log Z_0[\hat{\sigma}] + \log \langle e^{-V-V_h} \rangle_0 [\hat{\sigma}]$$

in the linear order in expansion the V_h term contributes $-\langle V_h \rangle_0 [\hat{\sigma}]$. For each red triangle the contribution

$$-\langle v_h \rangle_0 = h \left[\langle \sigma_1 \rangle_0 + \langle \sigma_2 \rangle_0 + \langle \sigma_3 \rangle_0 \right] (\hat{\sigma})$$

$$= 3h\, g(J_0)\, \hat{\sigma}$$

where we used our earlier result

$$\langle \sigma \rangle_0 = \underbrace{\frac{e^{3J_0} + e^{-J_0}}{e^{3J_0} + 3e^{-J_0}}}_{g(J_0)} \hat{\sigma}$$

Altogether, the new Hamiltonian in One decimation step (Keeping linear order)

Altogether, the new Hamiltonian in one decimation step (Keeping linear order)

$$\hat{H}[\hat{\sigma}] = -J' \sum_{\langle ij \rangle} \hat{\sigma}_i \hat{\sigma}_j - h' \sum_i \hat{\sigma}_i + \text{Constant}$$

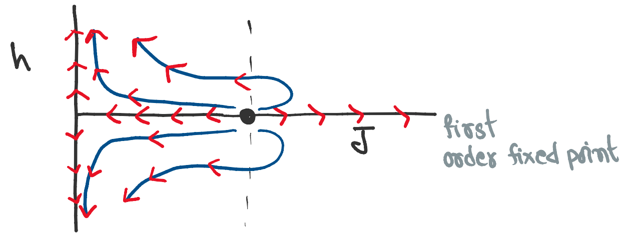
preserves the interactions, but only the couplings $(J, h) \longrightarrow (J', h')$ with

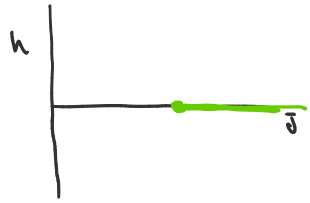
$$\boxed{\begin{aligned} J' &= R_1(J) = 2J\, g(J)^2 \\ h' &= R_2(J,h) = 3h\, g(J) \end{aligned} \qquad \text{with } g(J) = \frac{1+e^{-4J}}{1+3e^{-4J}}}$$

This is the RG equation

Flow diagram

To understand: for $J=0$, $h' = \frac{3}{2} h \Rightarrow$ fixed point at $h \to \pm\infty$. Not a line of fixed point unlike in 1d exact RG





Remarks: (1) $h=0$ line is invariant manifold, means, a poin originated on this line (manifold) remains on this manifold under RG transfomation.

(2) The $J \to \infty$ at $h=0$ fixed point is an attractive fixed point along $h=0$ line.

Critical behavior near $(J, h) \equiv (J^*, 0)$

(1) First we calculate linear stability in J and h directions

$$\delta J' = R_1'(J^*)\, \delta J \qquad \text{and} \qquad \delta h' = \left.\frac{\partial R_2(J,h)}{\partial h}\right|_{(J^*,0)} \delta h = 3\, g(J^*)\, \delta h$$

$$\boxed{\Rightarrow \delta J' = b^{y_1} \delta J \quad \text{with } y_1 \simeq 0.88}$$

$$\boxed{\Rightarrow \delta h' = b^{y_2}\, \delta h \quad \text{with } y_2 = \frac{\log \frac{3}{\sqrt{2}}}{\log \sqrt{3}} \simeq 1.37}$$

(2) the above two dependencies means

$$\delta J \sim b^{y_1}, \; \delta h \sim b^{y_2} \quad \Rightarrow \quad \delta h \sim (\delta J)^{y_2/y_1}$$

(3) The free energy density

$$f(J', h') = b^d f(J, h) + f_0$$

implies that leading non-analyticity of free energy density has singular dependence

$$f(\delta J', \delta h') = b^d f(\delta J, \delta h)$$

implies that leading non-analyticity of free energy density has singular dependence

$$f_{na}(\delta J', \delta h') = b^d f_{na}(\delta J, \delta h)$$

From scaling theory, the non-analyticity has a scaling form

$$\boxed{f_{na}(\delta J, \delta h) = (\delta J)^s \, \psi\left(\frac{\delta h}{(\delta J)^{y_2/y_1}}\right)} \Longrightarrow \boxed{s = d/y_1}$$

[You can alternately construct the scaling form by repeatedly applying decimation

$$f_{na}(b^{y_1}\delta J, b^{y_2}\delta h) = b^d f_{na}(\delta J, \delta h)$$

$$\Rightarrow f_{na}(b^{n y_1}\delta J, b^{n y_2}\delta h) = b^{nd} f_{na}(\delta J, \delta h)$$

Then choosing an n such that $b^{n y_1}\delta J \simeq 1$ gives

$$f_{na}\left(1, \frac{\delta h}{(\delta J)^{y_2/y_1}}\right) = (\delta J)^{-d/y_1} f_{na}(\delta J, \delta h)$$]

(4) From the scaling form of the free energy we get

$$m = \frac{\partial f}{\partial h} \sim (\delta J)^{s - \frac{y_2}{y_1}} \psi'\left(\frac{h}{\delta J^{y_2/y_1}}\right)$$

and

$$\chi = \left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} = (\delta J)^{s - 2\frac{y_2}{y_1}} \psi''(0) \Rightarrow \chi \sim \frac{1}{(\delta J)^{\gamma}}$$

with $\gamma = 2\frac{y_2}{y_1} - s = \frac{2y_2 - d}{y_1}$

similarly

$$m\Big|_{h=0} \sim (\delta J)^{\beta} \quad \text{with } \beta = s - \frac{y_2}{y_1} = \frac{d - y_2}{y_1}$$

alternatively

$$m \sim (\delta h)^{\frac{y_1}{y_2}\left(s - \frac{y_2}{y_1}\right)} \psi'\left(\frac{\delta h}{\delta J^{y_2/y_1}}\right)$$

$$\sim (\delta h)^{\frac{d - y_2}{y_2}} \phi\left(\frac{\delta J}{\delta h^{y_1/y_2}}\right) \Rightarrow m\Big|_{\delta J = 0} \sim (\delta h)^{1/\delta} \quad \text{with } \delta = \frac{y_2}{d - y_2}$$

A summary of all critical exponents

$$\alpha = 2 - \frac{d}{y_1}$$

$$\beta = \frac{d - y_2}{y_1}$$

$$\gamma = \frac{2y_2 - d}{y_1}$$

All exponents are expressed in terms of y_1 and y_2 which comes from linear stability of RG flow around the fixed point.
It affirms that there are only two independant scaling exponents
It gives aswell a scaling form

$$\gamma = \frac{2y_2 - d}{y_1}$$

$$\delta = \frac{y_2}{d - y_2}$$

$$\nu = \frac{1}{y_1}$$

It gives aswell a scaling form

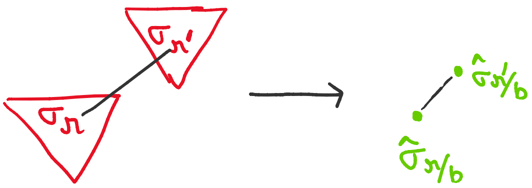
$$f_{na}(\delta J, \delta h) \sim (\delta J)^{\delta} \, \psi\left(\frac{\delta h}{\delta J^{y_2/y_1}}\right)$$

Moreover, we can form scaling relations

$$\alpha + d\nu = 2 \;\; ; \;\; \alpha + 2\beta + \gamma = 2$$

How does correlation scale under RG?

A handwaving argument



$$\langle \sigma(\bar{r})\sigma(\bar{r}')\rangle_c = G(\bar{r}-\bar{r}' \,|\, J, h)$$

$$\left\langle \hat{\sigma}\left(\frac{\bar{r}}{b}\right)\hat{\sigma}\left(\frac{\bar{r}'}{b}\right)\right\rangle_c = \hat{G}\left(\frac{\bar{r}-\bar{r}'}{b} \,\Big|\, J', h'\right)$$

Near critical point $\quad m \sim b^{d-y_2}$. It means, in one decimation step near critical point, m picks up a factor b^{d-y_2}

$$\Rightarrow \left\langle \hat{\sigma}\left(\frac{\bar{r}}{b}\right)\right\rangle \simeq b^{d-y_2}\langle \sigma(\bar{r})\rangle$$

$$\Rightarrow \left\langle \hat{\sigma}\left(\frac{\bar{r}}{b}\right)\hat{\sigma}\left(\frac{\bar{r}'}{b}\right)\right\rangle_c \simeq b^{2(d-y_2)}\langle \sigma(\bar{r})\sigma(\bar{r}')\rangle_c$$

$$\Rightarrow \boxed{\hat{G}\left(\frac{\bar{r}-\bar{r}'}{b}\right) \simeq b^{2(d-y_2)} G(\bar{r}-\bar{r}')}$$

In our linear order NvL method, the model remains same, and this means

$$\hat{G}\left(\frac{\bar{r}-\bar{r}'}{b}\right) = \boxed{G\left(\frac{\bar{r}-\bar{r}'}{b} \,\Big|\, J', h'\right) = b^{2(d-y_2)} G(\bar{r}-\bar{r}' \,|\, J, h)}$$

[For a clean derivation, see ch 9.8 of Goldenfeld]

Near critical point, from the linearized RG equation $\delta J' = b^{y_1}\delta J$ and $\delta h' = b^{y_2}\delta h$, it is expected that

$$G\left(\frac{r}{b} \,\Big|\, b^{y_1}\delta J, b^{y_2}\delta h\right) = b^{2(d-y_2)} G(r \,|\, \delta J, \delta h) \quad \text{[we exploited the notation]}$$

Repeating this RG steps n-times and choosing n such that $b^{ny_1}\delta J \simeq 1$ we get

$$G\left(r \cdot \delta J^{1/y_1} \,\Big|\, 1, \frac{\delta h}{\delta J^{y_2/y_1}}\right) = \frac{1}{\delta J^{\frac{2(d-y_2)}{y_1}}} \cdot G(r \,|\, \delta J, \delta h)$$

defining

$$\left(r \cdot \delta J^{1/y_1}\right)^{-2(d-y_2)} S\left(r \cdot \delta J^{1/y_1}, \frac{\delta h}{\delta J^{y_2/y_1}}\right)$$

we get

$\left(r \cdot \delta J^{1/y_1}\right) \qquad S\left(r \cdot \delta J^{1/y_1}, \frac{\delta h}{\delta J^{y_2/y_1}}\right)$

we get

$$\boxed{G\left(r \mid \delta J, \delta h\right) \simeq \frac{1}{r^{2(d-y_2)}} S\left(r \cdot \delta J^{1/y_1}, \frac{\delta h}{\delta J^{y_2/y_1}}\right)}$$

The above scaling form could be written in a more familiar for by using our earlier result for correlation length

$$\xi \sim \frac{1}{(\delta J)^{1/y_1}}$$

It gives

$$\boxed{G\left(r \mid \delta J, \delta h\right) \simeq \frac{1}{r^{2(d-y_2)}} S\left(\frac{r}{\xi}, \frac{\delta h}{\xi^{y_2}}\right)} \sim \frac{S(0,0)}{r^{2d-2y_2}} \quad \text{at critical point.}$$

It is customary to write as

$$\boxed{G(r) \simeq \frac{1}{r^{d-2+\eta}} S\left(\frac{r}{\xi}, \frac{\delta h}{\xi^{y_2}}\right)} \quad \text{with } \eta = d+2-2y_2$$

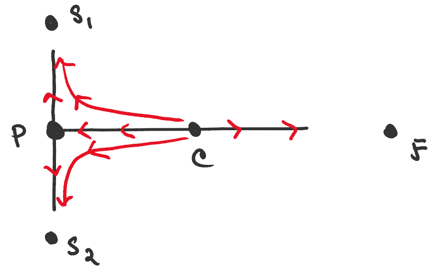
[$\theta = \frac{\eta}{2}$ is the anomalous dimension see ch7 of Goldenfeld]

Scaling relation.

using $\gamma = \frac{2y_2 - d}{y_1}$, $\nu = \frac{1}{y_1}$, and $\eta = d+2-2y_2$ we get another scaling relation

$$\boxed{\gamma = \nu(2-\eta)}$$

Other fixed points (non-critical)



at P: $\quad \delta J' = \left(\frac{1}{2}\right)\delta J \; ; \; \delta h' = \left(\frac{3}{2}\right)\delta h$

attractive along J direction,
repulsive along h direction

at F: setting $J \to \infty$ in the decimation equation, we see

$J' = 2J$ thus attractive along J direction.
$h' = 3h$ repulsive along h direction.

Remark: RG flows can also be thought as relation between systems at different points on coupling (J,h) space.

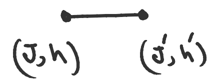
For example: $\quad f(J', h') = b^d f(J,h)$

$\Rightarrow$ magnetisation $\quad m(J',h') = \dfrac{\partial f(J',h')}{\partial h'}$

$$= b^d \cdot \frac{\partial f(J,h)}{\partial h'}$$

$$= b^d \cdot \frac{\partial h}{\partial h'} \cdot \frac{\partial f(J,h)}{\partial h} \longrightarrow m(J,h)$$

$$\Rightarrow \boxed{m(J,h) = b^{-d} \cdot \frac{\partial h'}{\partial h} \cdot m(J',h')}$$



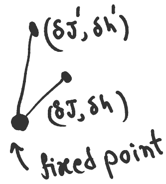
An important consequence of the above relation is that it allows us to see the nature of transition (Nienhuis-Nauenberg criteria).

For example consider the $J \to \infty$, $h = 0$ fixed point.

Around this point $\quad \delta J' = 2\delta J$ and $\delta h' = 3\,\delta h$ $\quad$ [ambiguity about $J \to \infty$ can be handled by considerin $1/J$ variable]

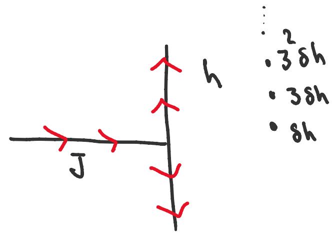
$$m(\delta J, \delta h) = b^{-d} \cdot 3 \cdot m(\delta J', \delta h')$$

$$= \tfrac{1}{\cancel{3}} \cdot \cancel{3} \cdot m(2\delta J, 3\delta h)$$



Repeating n-times

$$m(\delta J, \delta h) = m(2^n \delta J, 3^n \delta h)$$



Consider $\quad J \to \infty \Rightarrow \delta J = 0$

$$\Rightarrow \quad m(0, \delta h) = m(0, 3^n \delta h) \xrightarrow[n\to\infty]{} \pm 1$$

depending on sign of δh

this holds for $\delta h \to \pm 0$ and shows that m changes discontinuously with sign of h $\Rightarrow$ a first order transition.

Notice, how $m \to m$ is used and it came from $\boxed{\dfrac{\partial h'}{\partial h} = \Lambda_2 = b^d}$.

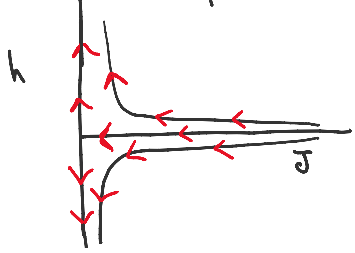
In general, if for a fixed point if for a coupling constant J, corresponding $\dfrac{\partial J'}{\partial J} = b^d$ (eigenvector) then there is a first order transition. This is the Nienhuis-Nauenberg criteria.

For a contrasting example, consider the $J = 0$, $h = 0$ fixed point

here

$$m(\delta J, \delta h) = 3^{-1} \cdot \tfrac{3}{2}\, m\left(\tfrac{1}{2}\delta J, \tfrac{3}{2}\delta h\right)$$

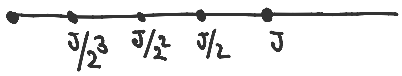
$$\Rightarrow \boxed{m(J,h) = \tfrac{1}{2}\, m\left(\tfrac{J}{2}, \tfrac{3h}{2}\right)}$$



$$\Rightarrow \quad \boxed{m(J,h) = \frac{1}{2}\, m\left(\frac{J}{2}, \frac{3h}{2}\right)}$$

along h=0 line:

$$m(J,0) = \frac{1}{2^n}\, m\left(\frac{J}{2^n}, 0\right)$$



because m is bounded, using large n,

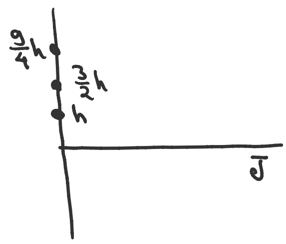
$m(J,0) \to 0$ thus confirms that $m \to 0$ as this fixed point is approached

(Caution: the analysis is only near the fixed point)

along J=0 line:

$$m(0,h) = \frac{1}{2^n}\, m\big(\underbrace{0, (3/2)^n h}_{\text{bounded}}\big)$$

$$\longrightarrow 0$$



therefore, no phase transition!

" The actual process of explicitly constructing a useful renormalization group is not trivial" ..... Michael Fisher

In general, one must be very careful about constructing an RG transformation. It is easy to get wrong results by not properly respecting symmetries of the problem.

For example: O(3) model or Heisenberg model.

$$H_{Heis} = -J \sum_{\langle ij \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j \qquad \text{with } \vec{\sigma} \equiv \text{a 3 component unit vector.}$$

Consider a decimation

$$\hat{\sigma} = \text{sign}\Big(\sum_{\text{block}} \sigma\Big) = \pm 1$$

This would map

$$H_{Heis} \longrightarrow H_{Ising}$$

This is wrong! It does not preserve O(3) symmetry.

Another example: Ising antiferromagnet. $(J<0)$

Our decimation step $J \to J'$ gives $J'>0$ thus ferromagnet!

The reason is, antiferromagnetic order is characterized by $q = \pi/a$ wavevector (Neil order), whereas RG picks up the $q \approx 0$ long wave vectors.

Ref: Leeuwen, PRL,(1975), 34, 1056.